Shaken sand, stress and test particles

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Abstract :

The definition of a stress tensor in shaken granular materials is difficult because the local properties change much in space and time. It is shown that in a stationary regime a definition of a standard form can be given at least in the case of a simple geometry (parallelepiped container, infinite along two dimensions, no symmetry breaking). In that case the stress is shown to be uniform. The quantities P^+ and P^- introduced by Evesque are found to be relevant to describe the force acting on a heavy test particle.

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1. Introduction :

The problem of interest in this article is defined for instance by Barrat et al. [1] as follows: "A granular gas is typically obtained by enclosing sand or balls made of glass, steel, brass, ceramic beads, etc in a container, which is subsequently vigorously shaken. The energy injected at the boundaries compensates for the dissipative collisions [7], and allows the grains (particles) to follow ballistic trajectories between collisions." The loss of kinetic energy at each collision is typically between 10 and 20 % [2]. The attention will be concentrated on a box infinite in two directions and shaken in the third direction z (Figure 1). This is consistent with the simulations of Herbst et al. [3]. It is assumed that no symmetry breaking structure appears, so that a stationary state is created, in which the density, the average velocity, the energy density, etc. are only functions of z. In certain cases, very complicated patterns have been observed [4], but these cases are excluded here.

2. The stress tensor

As pointed out by Evesque [5], the possibility of defining a stress tensor is not clear. A standard definition of the stress tensor is to consider the force exerted on a small volume element by the remainder of the fluid and write it as the divergence of a tensor, the stress tensor. However, in the systems of interest, the density can change appreciably in a distance comparable with the interatomic distance, and on such a distance the probability that a particle does not meet any other particle is high, so that such a particle does not exert any force at all.

I shall argue that, in the stationary regime and in the simple geometry of Figure 1, it is possible to define a stress tensor in a standard way.

I shall only consider the zz component of the stress tensor, which will be called $\sigma_{zz} = P$. Traditionally, it is deduced from the force exerted by the particles on one side of a plane z = Const on particles on the other side. If the force across an area S is F, then

$$\sigma_{zz} = P = F/S \qquad (1)$$

The difficulty is that particles are constantly going from one side to the other side. The definition (1) must therefore be made more precise. Each particle going from side A to side B during time t receives a label i. The component along z of its velocity is v_i. If t is long enough, the particle comes back with a velocity whose component along z is v'_i. The z-component of the force $F_i(t)$ of the force which has been acting on it satisfies

$$\int_{t_1}^{t_2} F_i(t)\, dt = m_i [v_i - v'_i] \qquad (2)$$

where t_1 and t_2 are the times where the particle has crossed the plane. We are only interested by events which satisfy $0 < t_1 < T$. At $t_2<T$ the particle loses its label, so that if $t_2<T$, (2) can be rewritten as

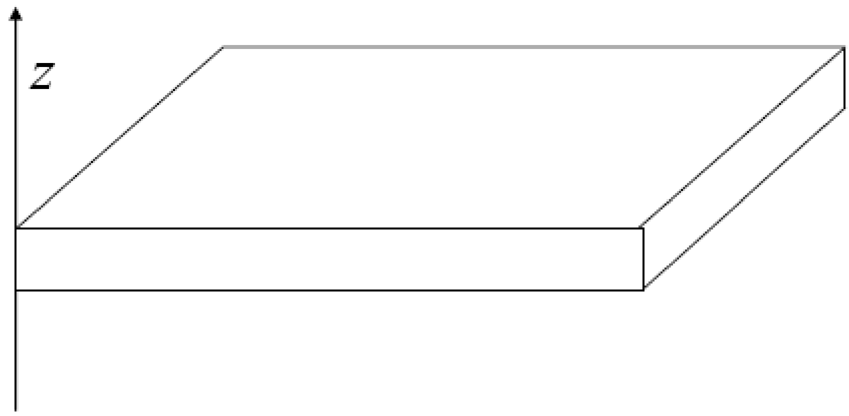

Figure 1: *The geometry*

$$\int_{0}^{T} F_i(t)\, dt = m_i [v_i - v'_i] \qquad (3)$$

If $t_2 > T$, (3) will be accepted too. The relative error is small if T is large.
It is now natural to define the total force F acting on one side of the plane by

$$F\,T = \Sigma_i\, m_i [v_i - v'_i] \qquad (4)$$

where the sum is over all particles which have crossed the plane from side A to side B during time T. They do not necessarily come back during time T, but if T is long enough one can replace (4) by

$$F\,T = \Sigma_i\, m_i\, v_i - \Sigma_r\, m_r\, v_r \qquad (5)$$

where r labels particles which have crossed the plane from side B to side A during time T. All v_i's have the same sign (the plus sign will be chosen), and the v_r's have the opposite sign.

In the stationary regime, the stress P defined by (1) and (5) is the same everywhere. In an usual fluid it is just the condition for mechanical equilibrium. Here it may be worth giving a proof, and this is done in Section 5.

Assuming all particles to have the same mass m, equation (5) can be written as

$$F\,T = \int_0^{+\infty} v\, n(v)\, dv \; - \; \int_{-\infty}^{0} v\, n(v)\, dv \qquad (6)$$

where n(v) is the number of particles whose z-component v_z of the velocity is v which cross the area S of the plane during time T. Its value is $n(v) = S\,\rho\, p(v)\, |v|\, T$ where ρ(z) is the local density and p(v) is the local probability density that $v_z = v$. The function p(v) depends on z as a parameter and, strictly speaking, should better be written p(z; v). It satisfies $\int_{-\infty}^{+\infty} p(z; v)dv = 1$. Formula (6) may be written as

$$F\,T = S\,T\,\rho \int_{-\infty}^{+\infty} v^2\, p(v)\, dv$$

so that the stress (1) is

$$P = \int_{-\infty}^{+\infty} v^2\, p(v)\, dv = \rho\, \langle v^2 \rangle \qquad (7)$$

This is the final formula of this section. The particle density ρ and the mean square velocity $\langle v^2 \rangle$ depend on z.

3. An example: the ideal gas

Formula (7) is a way to derive the equation of state of an ideal gas. Then, P is the pressure, ρ= N/V where N is the number of particles and V is the volume, and $p(v) = C \exp(-\beta m v^2)$, where $\beta = 1/(k_B\theta)$, k_B is the Boltzmann constant, θ the temperature, and C a constant determined by the condition $\int_{-\infty}^{+\infty} p(z; v)\, dv = 1$. The integration is readily performed and (7) yields the well-known formula

$$PV = N\, k_B \theta \qquad (8)$$

4. The Evesque pressure

Evesque [5] has introduced two quantities P^+ and P^- which have the dimension of a pressure and which, in the geometry of Figure 1, may be defined in analogy with (7) as

$$P^- = \rho \int_{-\infty}^{0} v^2\, p(v)\, dv \qquad (9)$$

$$P^+ = \rho \int_{0}^{+\infty} v^2\, p(v)\, dv \qquad (10)$$

In an ideal gas at equilibrium, both quantities are equal to half the pressure. In a shaken sand experiment, these « Evesque pressures » are clearly different; P^- is not equal to P^+. This is related to the strong asymmetry of the velocity distribution function p(z; v), at least near the vibrating walls (Figure 2). Evesque calls $-P^-$ what is called P^- here.

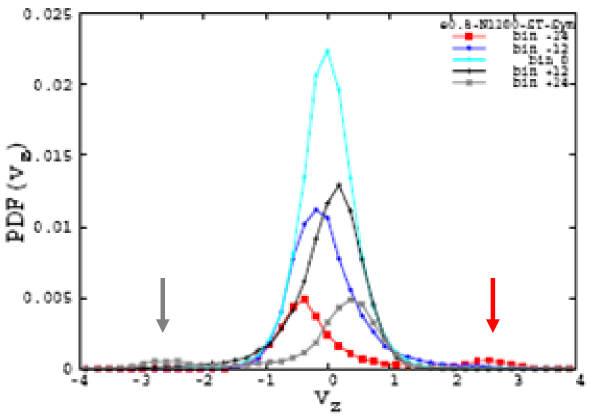

Figure 2: *Velocity distribution in different places. Near a vibrating wall, it has two peaks (red curve). Taken from ref. [2]*

The Evesque pressure carries more information than the «standard» stress defined by (7). Moreover, it has a physical meaning. Indeed, it characterizes the force acting on a very heavy test particle of vanishing or small velocity inserted in the gas of other particles. Consider a collision between the test particle of momentum **P** and a light particle of momentum **p**. After the collision, the momenta P' and p' satisfy **P'**- **P**= **p**- **p'**. The modulus of this quantity is of the order of **p**. Near a vibrating wall, the velocity distribution is strongly asymmetric, with two peaks (Figure 2). Near one of the walls, particles with a positive velocity are much faster than those with negative velocity. Near the other wall, it is the contrary. On the other hand, the number of particles meeting the test particle from the +z direction is the same as the number of particles meeting the test particle from the -z direction. This results from the equality.

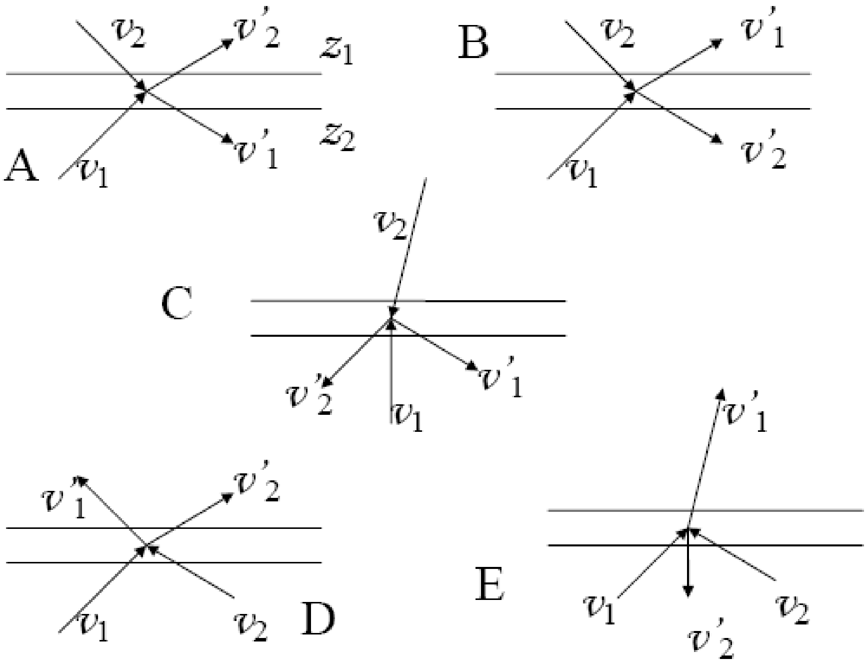

Figure 3: *Various types of collisions. The z-component of the velocities are denoted* v_1 *and* v_2 *(incoming particles),* v'_1 *and* v'_2 *(outgoing particles).*

Near a vibrating wall, because of the asymmetric velocity distribution, $|p_z|$ is, on the average, much larger for the particles which push the particles away from the wall than for particles which push it to the wall. On the other hand, the number of particles meeting the test particle in the same time is the same for both types of particles. This results from the equality

$$\int_{-\infty}^{+\infty} v\, p(v)dv = 0 \qquad (11)$$

valid in the stationary regime because the average velocity is zero everywhere. Therefore, the test particle is pushed away from the walls. A detailed argument (which will not be presented here) suggests indeed that is undergoes a force

$$F = \varpi\, [P^+ - P^-] \qquad (12)$$

Where ϖ has the dimension and the order of magnitude of the cross section of the test particle (i.e. πr^2 for a spherical particle of radius r). Its precise form depends on details, such as the degree of elasticity of collisions). Formula (12) holds if the velocity of the test particle is small with respect to that of the other particles.

Quantities P^+ and P^- can be determined numerically or experimentally [2], and it is clear that the force (12) is directed towards the middle of the container. Thus, formula (12) suggests that in the stationary regime, heavy particles are confined in the central part of the container.

Evesque [5] has suggested the same effect for big particles. However, a more precise analysis may be necessary because such particles modify the density in their neighbourhood. For instance a big particle near a vibrating wall projects a shadow on the wall, which does not receive particles and therefore does not reflect them. Therefore a big particle near a vibrating wall might be projected against the wall by particles coming from the opposite side.

One can also consider the case when the « test particle » is identical to the other particles. Then, after a few collisions, the average force acting on it is the same as that acting on the other particles which is 0. Indeed, the average position $< z >$ of all particles is the same, time-independent quantity (equal to 0 if the container is centred at $z = 0$). Therefore the average acceleration $< d^2z/dt^2 >$ vanishes and the average force too. Of course, as well-known, the average velocity vanishes too, this is formula (11).

We now come back to the "standard" stress. It has a fundamental property which makes it useful: it is uniform in the stationary regime, as will now be shown.

5. Uniformity of the stress

This section, as the preceding ones, is devoted to the shaken box of Figure 1, infinite in two directions, in the stationary regime. It may look intuitive that stationarity implies mechanical equilibrium and therefore uniformity of the stress. However, in view of the permanent exchange of particles between the various parts of the system, a detailed derivation is useful.

Let z_1 and z_2 be the positions of two planes in the box. We want to show that the stresses $P(z_1)$ and $P(z_2)$ defined above are equal.

Any possible difference between $P(z_1)$ and $P(z_2)$ would be a result of collisions between particles between z_1 and z_2. These heights will be assumed nearly equal, so that multiple collisions are excluded. For instance a particle 1 with velocity $v_1 > 0$ can collide with a particle 2 with velocity $v_2 < 0$. After the collision, the particles have velocities v'_1 and v'_2 which satisfy momentum conservation, namely

$$v_1+v_2 = v'_1+v'_2 \qquad (13)$$

Four cases are possible (Figure 2), namely

A. After the collision, $v'_1 < 0$ and $v'_2 > 0$
B. After the collision, $v'_1 > 0$ and $v'_2 < 0$
C. After the collision, $v'_1 < 0$ and $v'_2 < 0$
C'. After the collision, $v'_1 > 0$ and $v'_2 > 0$

In case A, the contribution $\delta F_1 T$ of particles 1 and 2 to (5) is $m_1(v_1 - v'_1)$ at height z_1 and the contribution $\delta F_2 T$ of the same particles at height z_1 is $m_2(v_2 - v'_2)$. Both contributions are equal according to (13). Expressions of $\delta F_1 T$ and $\delta F_2 T$ in cases B and C are shown in table 1 and are also equal according to (13). Case C' is clearly analogous to case C.

A collision between two particles with both positive velocities v_1 and v_2 is also possible. Two possibilities D an E are to be considered (Figure 2). In both cases, the contributions $\delta F_1 T$ and $\delta F_2 T$ are given in table 1 and found to be equal according to (13). This demonstrates that the stress is uniform in the stationary regime and in the geometry of Figure 1.

process	**A**	**B**	**C**	**D**	**E**
Cont. to F_1T *Cont.* to F_2T	$m_1(v_1-v'_1)$ $m_2(v_2-v'_2)$	$m_1v_1 - m_2 v'_2$ $m_1v'_1 - m_2 v_2$	$m_1v_1-m_1v'_1- m_2 v'_2$ $-m_2v_2$	$m_1v_1 +m_2 v_2$ $m_1v'_1 + m_2 v'_2$	$m_1v_1+ m_2v_2 - m_2 v'_2$ $m_1v'_1$

Table 1: *Contributions to* F_1T *and* F_2T

6. Dynamics and other extensions

Can the concept of stress be used in a situation which is not stationary? This is questionable. In usual hydrodynamics, matter is assumed to be in local equilibrium. This means that in a sufficiently small volume the density, the energy density, etc. are approximately uniform in space and time. In shaken sand this is hardly possible: if the density is uniform, the volume should be so small that strong fluctuations in time are expected. The time-dependent behaviour might require solving the Boltzmann equation without any possible simplification. For instance it does not seem possible to write a Navier-Stokes equation.

In contrast with the standard stress or pressure the « Evesque pressures » P^+ and P^- can clearly be useful in a time-dependent situation. For instance formula (12) describes the force acting at a given time on a test particle while (2) does not, since particle i may have a long history between incoming with velocity v_1 and outgoing with velocity v_2.

Even in the stationary case, the definition of a stress tensor is not straight forward if the container has not the simple shape of Figure 1. The case of an hourglass [5], where the flow is nearly interrupted by shaking, is particularly appealing. The problem is presumably not extremely difficult; it is left for further research by other researchers.

7. Conclusion

The present work was largely motivated by Pierre Evesque's work, criticizing the very concept of pressure and stress tensor in shaken sand and advocating the use of asymmetric quantities P^+ and P^-. I have shown, in a simple case, that the concept of stress tensor can be used in a stationary regime, but is otherwise questionable. On the other hand, Evesque's `pressures' P^+ and P^- are physically meaningful quantities which can be used in any circumstance, even not stationary. I hope everybody will be happy with this diplomatic conclusion.

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